JOURNAL OF NATURAL GEOMETRY, 16(1999)125-140 **0963-2654-1999-16-9-125-140**
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The Quaternionic Structure of 3-dimensional Natural Geometry

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Abstract. An isomorphism between EUCLIDean triangles and HAMILTON'S quaternions is derived. Two parameters, *c* and *h*, appear in this isomorphism and we will interpret them physically as *velocity of light* and PLANCK'S *constant.* The parameters *c* and *h* are constrained by the equation ch = i, and this leads to the need to introduce imaginary physical metrical units.

1. Triangles and quaternions

Quaternions, elements of an algebraic skew field, the HAMILTONian form of complex numbers are traditionally taken as points of a *4-dimensional* space, the four basic vectors 1, *i*, *j*, *k* of this space possess the multiplication structure

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

(details in for example [8]).

HAMILTON had much trouble to construct his numbers for he could not find a point (or vector) picture of his number elements in the *3-dimensional* space of our visual perception. Such a model does not exist. But one gets a 3-dimensional picture of our number field if we perceive its elements not as points or vectors but as EUCLIDean triangles.

In the following Sections we will find an isomorphism between quaternions and 3-dimensional triangles.

Let V be the 3-dimensional EUCLIDean vector-space. Now we consider pairs (a_1, a_2) of vectors. If we take a fixed point M of 3-dimensional space, we can identify (a_1, a_2) with the triangle (M, $M + a_1$, $M + a_2$). (In this sense, the triangles (A, B, C) and (A, C, B) are different of course.) Because of this identification, we will often call (a_1, a_2) a triangle. We, will call $\mathbf{T} = \mathbf{V} \times (\mathbf{V} - \{\mathbf{0}\})$ the set of fundamental triangles.

Definition 1.1.
Let (a_1, a_2), (b_1, b_2) $\in$ **T***; the pairs* (a_1, a_2) *and* (b_1, b_2) *are equivalent, if*

$$\frac{(a_1 a_2)}{|a_2|^2} = \frac{(b_1 b_2)}{|b_2|^2}$$
$$\frac{a_1 \times a_2}{|a_2|^2} = \frac{b_1 \times b_2}{|b_2|^2} \qquad (1.1)$$

(a_1, a_2) *is the inner-*, $a_1 \times a_2$ *the outer-product of the vectors* a_1 *and* a_2*.*

In symbols:

$$(a_1,a_2) \sim (b_1,b_2)$$

(This is of course an equivalence relation, so we have **T**/~ as the set of all classes of equivalent vector pairs.) We can transfer this definition to our interpretation of the vector pair by a triangle. It is easy to verify from equation (1.1) that two triangles are equivalent, if and only if they are parallel and similar.

In other words: If we take the triangle (a_1, a_2), rotate it around the axis $a_1 \times a_2$ yielding the triangle (a_1', a_2') and stretch it by a factor $\lambda \neq 0$, then $(a_1, a_2) \sim (\lambda a_1', \lambda a_2')$. We should as well note that all triangles (0, a_2) are equivalent as well.

We will now need a pair (c, h) of complex numbers with

$$ch = \mathrm{i} \qquad (1.2)$$

We call these constants c and h the *fundamental metrical constants.* Of special interest are the cases (c, h) = (i, 1) and (c, h) = (1, i) as we shall see later, but in fact none of our proofs will be based on this special values.

We are now going to identify the factor structure **T**/~ with the quaternions by making use of the common mixed scalar-vector representation of quaternions:

Definition 1. 2.
Let (a_1, a_2), (b_1, b_2) $\in$ **T; ***we set*

$$\alpha := \frac{(a_1 a_2)}{|a_2|^2}$$
$$a := hc^2 \frac{a_1 \times a_2}{|a_2|^2} \qquad (1.3)$$

Then ($\alpha|a$) is *the correspondent quaternion number. We call* α *the* scalar part *and* a *the* vector part of *the quaternion.*

Again it is obvious that this assignment is consistent with our equivalence relation above, so we can assign one unique quaternion number to every equivalence class $[(a_1,a_2)]_\sim$ of $\mathbf{T}/\sim$. Vice versa we can construct a pair (a_1, a_2) for every quaternion $(\alpha|a)$, so that (1.3) holds (take a_2 to be a unit vector orthogonal to a, the rest is simple vector arithmetic). So we have:

Theorem 1.3.
There is a one-to-one correspondence between (*a subset of complex*) *quaternions and* $\mathbf{T}/\sim$, *such that for every* $[(a_1,a_2)]_\sim \in \mathbf{T}/\sim$ *equations* (*1.3*) *hold.*

Remarks.

- For simplicity we will write $[(a_1,a_2)]$ for $[(a_1,a_2)]_\sim$.

- As there are no constraints on c and h besides (1.2), the vector part of the quaternion will usually be a complex vector. It should also be noted that the quaternion corresponding to the fundamental triangle (a_1, a_2) *is* zero if and only if $a_1 = 0$; we will denote the equivalence class of these triangles with $[(0,e_1)]$.

- Only with the special norm $(c, h) = (\mathrm{i}, 1)$ we come back to the usual description of HAMILTONian quaternions. But our results do not depend on this special assumption. We describe the HAMILTONian skew field and its elements in a more general form. Since our triangles – c and h only being restricted by (1.2) – are elements of a skew field isomorphic to HAMILTONian quaternions, we will use the term 'quaternion' for the triangle numbers and their algebraic description without confusion.

- I call $(c, h) = (1, \mathrm{i})$ the *hyperbolic state* of HAMILTONian quaternions (cf. [5]).

2. Triangle composition and quaternionic product

We, will define an operation on our set **T**/~ and prove that this operation corresponds to the quaternionic product.

Given two equivalence classes of triangles $[(a_1,a_2)]$ and $[(b_1,b_2)]$, we will define the *composition* of these triangles as follows:

(a) If $a_1 \neq 0$ and $b_1 \neq 0$, then we can find triangles (a_1', a_2') and $(\mathrm{b}_1', \mathrm{b}_2')$ with $(a_1, a_2) \sim (a_1', a_2')$ and $(b_1, b_2) \sim (b_1', b_2')$ and $a_2 = b_1'$ (this is quite easy to see: Clearly the planes of the two triangles (a_1,a_2) and (b_1,b_2) must intersect if not be identical. So we only have to rotate the two triangles within their respective plane and line up a_2 and b_1 with the intersection line of the two planes and apply a proper stretching factor). Then we define $[(x_1,x_2)]$ with (x_1,x_2):= (a_2, b_1') to be the composition of $[(a_1,a_2)]$ and $[(b_1,b_2)]$. (Our assumptions assure $x_1 \neq 0$).

(b) If $a_1 = 0$ or $b_1 = 0$ then we define $[(0, e_1)]$ to be the composition.

First of all we will have to see that the composition is well defined. This is obvious in case (b), only case (a) might require some reflection to see that all different values of (x_1,x_2) left open in the definition lead to the same equivalence class $[(x_1,x_2)]$ (the triangle (x_1,x_2) as defined above is uniquely determined except for a stretching factor, and – if the triangles (a_1,a_2) and (b_1,b_2) lie within the same plane – a rotation in this plane; both stretching and rotation in this plane lead to the same equivalence class).

We will now show that the composition of $[(a_1,a_2)]$ and $[(b_1,b_2)]$ corresponds to the multiplication of the corresponding quaternions $(\alpha|a)$ and $(\beta|b)$. In case (b) there is nothing to prove, so we need only consider case (a): If we have $A = (\alpha|a)$ and $B = (\beta|b)$ with

$$\alpha = \frac{(a_1 a_2)}{|a_2|^2}$$

$$a = hc^2 \frac{a_1 \times a_2}{|a_2|^2}$$

$$\beta = \frac{(a_2 a_3)}{|a_3|^2}$$

$$b = hc^2 \frac{a_2 \times a_3}{|a_3|^2}$$

then we shall see, that the product $X = A \cdot B = (\xi|x)$ is described by

$$\xi = \frac{(a_1 a_3)}{|a_3|^2}$$

$$x = hc^2 \frac{a_1 \times a_3}{|a_3|^2}$$

Proof. We will make use of a slightly modified version of the well-known formula for the product of two quaternions, i.e.:

$$\begin{aligned} \xi &= \alpha\beta + \frac{(ab)}{c^2} \\ x &= \alpha b + \beta a + ha \times b \end{aligned}$$

(In the case c = i, h = 1 we find the usual version of this formula, cf. [7])

With a little vector arithmetic and (1.2) we find for the scalar part ξ:

$$\begin{aligned} \xi &= \frac{(a_1 a_3)}{|a_3|^2} = \frac{(a_1 a_3)(a_2 a_2)}{|a_2|^2 |a_3|^2} \\ &= \frac{(a_1 a_2)(a_2 a_3) - \big((a_1 a_2)(a_2 a_3) - (a_1 a_3)(a_2 a_2)\big)}{|a_2|^2 |a_3|^2} \\ &= \frac{(a_1 a_2)(a_2 a_3) - (a_1 \times a_2)(a_2 \times a_3)}{|a_2|^2 |a_3|^2} = \alpha\beta + \frac{(ab)}{c^2} \end{aligned}$$

For the vector part x, we set

$$\begin{aligned} z &:= \alpha b + \beta a + h \cdot a \times b \\ &= hc^2 \frac{(a_1 a_2)(a_2 \times a_3) + \big((a_1 \times a_2)(a_2 a_3) - (a_1 \times a_2) \times (a_2 \times a_3)\big)}{|a_2|^2 \cdot |a_3|^2} \\ &= hc^2 \frac{(a_1 a_2)(a_2 \times a_3) + (a_1 \times a_2)(a_2 a_3) - (a_1 a_2 a_3) a_2}{|a_2|^2 \cdot |a_3|^2} \end{aligned}$$

(where $(a_1 a_2 a_3)$ denotes the inner-product of the vectors a_1 and $a_2 \times a_3$), we have to show: $x = z$. We find:

$$xa_1 = 0, \qquad xa_3 = 0, \qquad xa_2 = -hc^2 \frac{(a_1 a_2 a_3)}{|a_3|^2}$$

It follows: $z = \lambda a_1 \times a_3$. Assuming, a_1, a_2 and a_3 are linearly independent, we find noting that $(a_1 a_2 a_3) \neq 0$:

$$\lambda (a_1 \times a_3) a_2 = -hc^2 \frac{(a_1 a_2 a_3)}{|a_3|^2}$$

$$\lambda = \frac{hc^2}{|a_3|^2}$$

$$z = hc^2 \frac{a_1 \times a_3}{|a_3|^2} = x$$

If a_1, a_2 and a_3 are linearly dependent, we may assume that

$$a_2 = \lambda_1 a_1 + \lambda_2 a_3$$

In this case $(a_1 a_2 a_3) = 0$ and we have:

$$\begin{aligned}
z &= hc^2 \frac{(a_1 a_2)(a_2 \times a_3) + (a_2 a_3)(a_1 \times a_2)}{|a_2|^2 \cdot |a_3|^2} \\
&= hc^2 \frac{\lambda_1 (a_1 a_2)(a_1 \times a_3) + \lambda_2 (a_2 a_3)(a_1 \times a_3)}{\left(\lambda_1^2 a_1^2 + 2\lambda_1 \lambda_2 (a_1 a_3) + \lambda_2^2 a_3^2\right) \cdot |a_3|^2} \\
&= hc^2 \frac{\left(\lambda_1^2 a_1^2 + 2\lambda_1 \lambda_2 (a_1 a_3) + \lambda_2^2 a_3^2\right)(a_1 \times a_3)}{\left(\lambda_1^2 a_1^2 + 2\lambda_1 \lambda_2 (a_1 a_3) + \lambda_2^2 a_3^2\right) \cdot |a_3|^2} \\
&= hc^2 \frac{a_1 \times a_3}{|a_3|^2}
\end{aligned}$$

This completes our proof which is summarized in the following theorem:

Theorem 2.1. *Let* $[(a_1,a_2)]$ *and* $[(b_1,b_2)]$ *be equivalence classes of triangles, A and B the corresponding quaternions, respectively; let* $[(x_1,x_2)]$ *be the composition of* $[(a_1,a_2)]$ *and* $[(b_1,b_2)]$. *Then the quaternion* $A \cdot B$ *corresponds to the triangle class* $[(x_1,x_2)]$

Remarks. The correspondence between composition of triangles and multiplication of quaternions has some amazing properties:

1. The triangle class $[(a,a)]$ with $a \neq 0$ represents unity, which can be seen by considering the effect of composition (or by equations (1.3)), $(0,a)$ is the zero element.

2. If $[(a_1,a_2)]$ with $a_1 \neq 0$ is an equivalence class of triangles and A the corresponding quaternion number, then the inverse A^{-1} is represented by $[(a_2,a_1)]$; furthermore we see that the quaternion corresponding to $[(a_1,a_2)]$ is invertible if and only if $a_1 \neq 0$ (that is, every quaternion except for the zero element is invertible).

3. The correspondence allows an easy proof of associativity of quaternion product: The two possible orders of composition of the three triangles (a_1,a_2), (a_2,a_3) and (a_3,a_4) obviously both lead to (a_1,a_4).

4. A similar construction (with less spectacular results) may be used to express addition of quaternions in terms of triangles: We can always transform the triangles (a_1,a_2) and (b_1, b_2) into the equivalent triangles (a_1', c) and (b_1', c) and then define the sum to be $(a_1' + b_1', c)$.

5. Composition of triangle classes is commutative if the triangle classes occupy the same plane ε. This leads to a commutative subset of quaternions isomorphic to the set of complex numbers. Moreover, if we take ε to be a fixed unit vector in ε then every quaternion of this set can be described as the equivalence class [(*a*,ε)] with *a* being a vector in ε; if we now identify the vector *a* with this equivalence class then we see that this naturally leads to the GAUSS plane of complex numbers.

6. Considering composition of triangles, we can easily see that our structure has no zero divisors.

7. Didactic aspects of representing numbers by triangles were discussed in [7].

Quaternions are traditionally classified as 'real' or 'complex' if the 4-dimensional vector – traditionally used to construct quaternions – has real or complex components.
HAMILTONian quaternions are usually identified with 'real' quaternions. From the triangle model we get the HAMILTONian quaternions with the two complementary states (*c*,*h*) = (i,1) and (c, *h*) = (1,i). The existence of the non-classical, hyperbolic state (*c*,h) = (1,i) raises the question: Is the above classification useful if the HAMILTONian quaternions possess *non-real* components in the hyperbolic state?

Some theoretical papers construct a relativistic quantum theory in using 'complex' quaternions (cf. [1, 2, 9, 10]), but they do not realize that HAMILTONian quaternions in its hyperbolic metrical representation do produce these results.

Some physical papers (cf. [3, 4]) discuss the geometrical meaning of quaternions, but do not realize that HAMILTONian quaternions are EUCLIDean triangles in the natural space of our perception.

Using the triangle model and our algebraic, complementary description of the quaternionic skew field, we get a 3-dimensional picture of MINKOWSKI'S space-time (cf. [6]) and a better connection between relativity and quantum mechanics.

3. Physical interpretations

We, will now try to show how our interpretation of quaternions can be applied in physics.

3.1. Lorentz transformation

If we assume (c,h) = (1,i), then we can deduce the formalism of special relativity as follows:

A *space-time event* (or kinematical event) is described by

$$Q = (t|x) = t + x$$

when the real part t describes the time, and the vector part x describes the location. A *material event* (or dynamical event) is described by

$$Q \to Q' = Q \cdot W \qquad \text{and} \qquad P \to P' = P \cdot W$$

within $W \cdot \overline{W} = 1$.

Special relativity possesses two basic invariants, namely

$$\text{proper time} \quad \tau := \sqrt{Q \cdot \overline{Q}}$$

and

$$\text{rest mass} \quad \mu := \sqrt{P \cdot \overline{P}}$$

Invariance of proper time and rest mass in the LORENTZ group follows immediately, for instance

$$Q' \cdot \overline{Q'} = (Q \cdot W) \cdot \left(\overline{Q \cdot W}\right) = (Q \cdot W) \cdot \left(\overline{Q} \cdot \overline{W}\right) = Q \cdot \left(W \cdot \overline{W}\right) \cdot \overline{Q} = Q \cdot \overline{Q} .$$

The skew field formalism produces a very compact description of special relativity. We get the LORENTZ transformation in the usual form if we separate in real and vector part:

$$W = \frac{1 + v}{\sqrt{1 - v \cdot v}} = \frac{1 + v}{\sqrt{1 - \frac{(vv)}{c^2}}}$$

Noting that $W \cdot \overline{W} = 1$ we get the formulas for the *Kinematical* LORENTZ *transformations*:

$$t' = \gamma \left(t + \frac{(xv)}{c^2} \right)$$

$$x' = \gamma(x + tv + hx \times v)$$

with the usual factor $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{(vv)}{c^2}}}$.

In the same way we can derive *Dynamic* LORENTZ *transformation*:

$$m' = \gamma\left(m + \frac{(pv)}{c^2}\right)$$
$$p' = \gamma(p + mv + hp \times v)$$

Our form of special relativity is possible only if we. accept imaginary metrical units for space vectors in space-time. This formalism also shows another difference to the common theory: The number *h*, representing PLANCK'S constant in microphysics, does not appear explicitly in the EINSTEINean theory. Our formalism reveals the terms

$$hx \times v \qquad \text{and} \qquad hp \times v$$

in the LORENTZ transformation (with a *pure* number *h*).

The common theories of relativity and quantum mechanics do not suppose this fundamental connection between velocity of light *c* and wirkungsquantum *h* as described by our fundamental metrical equation *ch* = i. A better foundation and correlation of both relativity and quantum mechanics is achieved if we choose a physical metrical system based on *ch* = i. We should also note that in classical quantum theory the wirkungsquantum *h* is strongly associated with i = $\sqrt{-1}$.

3.2. Dynamic events and DE BROGLIE waves

We gave a first physical interpretation of quaternions by writing LORENTZ transformations as functions in the skew field. We used quaternions to describe points $Q = (t|x)$ in space-time and impulses $P = (m|p)$ dynamically. We will now continue this interpretation. Let

$$m^* := \frac{(a_1 a_2)}{|a_2|^2}$$
$$p_* := \frac{a_1 \times a_2}{|a_2|}$$

We now interpret m^* as the scalar *energy of a motion,*

$$P^* := m^* + hc^2 p_* = c^2\left(\frac{m^*}{c^2} + hp_*\right)$$

as *quaternionic energy*,

$$P := \frac{m^*}{c^2} + hp_*$$

as the *quaternionic impulse*.

$$m := \frac{m^*}{c^2}$$

is the *mass* of the moved object and

$$p := hp_*$$

is the vector K. p_* is the DE BROGLIE *wave vector* of this motion. So *P* as defined above has the form

$$P = m + p$$

Since

$$P = \left(\frac{m^* / h}{c^2} + p_* \right) h$$

one may call

$$P_* := \frac{m^* / h}{c^2} + p_*$$

the *wave quaternion*.

$$\nu := \frac{m^*}{h}$$

is the *frequency* of the wave, so that the wave quaternion takes the form

$$P_* := \frac{\nu}{c^2} + p_* .$$

The connection between the three levels P^*, P and P_* ('energy'-, 'mass'- and 'frequency'-level) is

$$\frac{P^*}{c^2} = P = hP_* ,$$

expressed in its components:

$$m^* = mc^2 = h\nu$$
$$p = hp_* ,$$

leading to the basic energy equations

$$m^* = mc^2 \qquad \text{EINSTEIN}$$
$$m^* = h\nu \qquad \text{DE BROGLIE}$$

and the DE BROGLIE equation

$$p = hp_*$$

connecting the wave vector p_* and the vector impulse p. The natural reproduction of these classical physical equations support our hypothesis that the fundamental metrical parameters can be interpreted as velocity of light and PLANCK constant h.

3.3. A modified metrical system

Starting with the pure mathematical structure of quaternions we found several arguments for a physical interpretation of the fundamental metrical numbers c and h.

Now we will turn around and take the opposite way starting with a classical equation which expresses the connection of c, h with the electronic charge ε. We write the SOMMERFELD equation

$$\alpha = \frac{2\pi\varepsilon^2}{hc}$$

with the fine structure constant $\alpha \approx \frac{1}{137}$ as

$$(\alpha c)\left(\frac{h}{2\pi}\right) = \varepsilon^2$$

and conclude: All classical physical metrical systems assume

$$\delta_1 = \delta_2$$

with

$$\delta_1 := (\alpha c)\left(\frac{h}{2\pi}\right) \qquad \text{and} \qquad \delta_2 := \varepsilon^2$$

This axiom $\delta_1 = \delta_2$ is not the only possible one. Our hypothesis is: It is more natural to assume

$$\delta_1 = i \qquad \text{and} \qquad \delta_2 = 1$$

This leads to

$$\varepsilon = \pm 1$$

for the metrical number of the elementary charge, and to the fundamental metric equation for

$$\text{the velocity of light} \qquad \bar{c} := \alpha c$$

$$\text{the wirkungsquantum} \qquad \bar{h} := \frac{h}{2\pi}.$$

This metrical system transforms the SOMMERFELD equation in ε ± 1 together with our fundamental metrical equation

$$\bar{c} \cdot \bar{h} = i\,.$$

All our results remain valid if we substitute (c, h) by $(\bar{c}, \bar{h})$, as the only difference is a scalar factor. This modified metrical system opens the way to a quaternionic description of physics, with a better connection between relativity and quantum mechanics.

Our quaternionic triangles reduce the 4-dimensional world of space-time to a world of EUCLIDean triangles situated in the natural 3-dimensional space of our visual perception.

If we use the metrical system $(\bar{c}, \bar{h})$ = (1,i) for describing such space-time triangles, this hyperbolic quaternionic description of nature leads to the natural norms

$$\varepsilon = \pm 1$$

$$h = 2\pi i$$

$$c = \frac{1}{\alpha}$$

for elementary charge, wirkungsquantum and velocity of light.

On this way we may identify the elementary spin $\frac{1}{2}h$ of a particle with the EUCLIDean angle sum of every HAMILTONian quaternion.

Acknowledgements. The author wishes to thank B. Bosbach, C. Budzinski, S. Coya, G. Dobbrack †, E. Hultsch, W. Pfeiffer, A. Pomp, R. Ruthenberg †, A.Seery, H.Steinhaus for critical remarks, help and encouragement. I am deeply indebted to W. Sproessig (Freiberg). Mathematical parts of the paper were specified by H. Bahmann (Freiberg).

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RECEIVED BY THE EDITOR: MARCH 23, 1999.